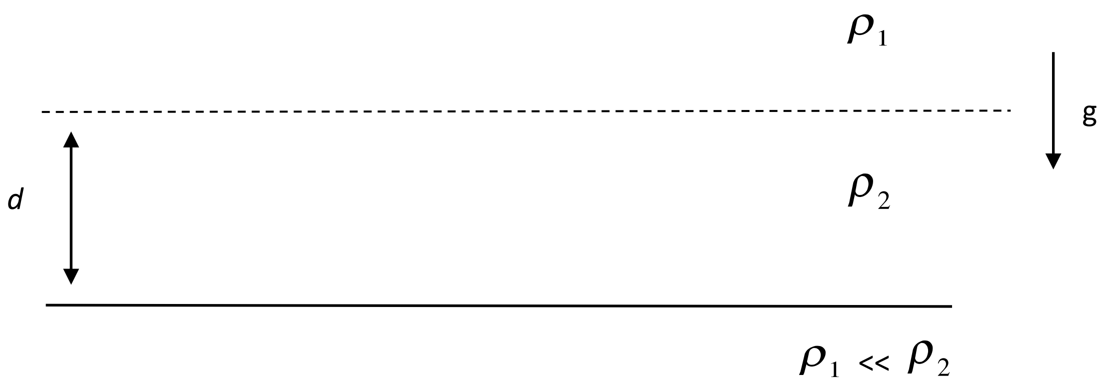
Problem Set III: Due TBA

1a) Determine the general dispersion relation for surface waves in a fluid of finite depth *d*. Treat the fluid as ideal.



b) Discuss the limits $kd >> 1$, $kd << 1$.

c) For $kd << 1$, deduce by analogy with sound waves the equations describing surface waves in shallow water. Hint: the dynamical fields are water height and horizontal velocity. Try to deduce/guess the nonlinear equations, called shallow water equations.

d) Comment on the relevance of shallow water dynamics to the objective of ripple tank demonstrations, frequently used to stimulate optical wave phenomena in high school physics classes.

2) In MHD, the Ohm's Law is

$$\underline{E}+\frac{\mathrm{v}\times\underline{B}}{c}=\eta\underline{J}$$

and displacement current is neglected (low frequency!), so – with Faraday's Law – one obtains the magnetic induction equation, which closely resembles the vorticity equation.

a) Derive the magnetic field induction equation. Show $\underline{B}/\rho$ is frozen-in for compressible ideal MHD.

b) For ideal MHD, prove Alfven's Theorem:

$$\oint_s \underline{B}\cdot d\underline{a}=const.$$

Be sure to treat motion of the loop. What is this the counterpart of?

c) What does Alfven's theorem mean?

3a) Derive the dispersion relation for buoyancy waves in a stably stratified fluid with $\partial S/\partial Z>0$ and $g=-g\tilde{z}$. These are called internal waves. Take the equilibrium hydrostatic. Show that internal waves are 'backward', i.e. the phase and group velocity can be in opposite directions.

b) Generalize your analysis of internal waves to include rotation effects, where $\Omega=\Omega\hat{z}$. When are corrections to the dispersion relation due to rotation of significance?

4) Falkovich observes that the interfacial version of the ideal flow shear driven instability (i.e. the Kelvin-Helmholtz instability) necessarily has a maximum (or minimum) in the profile of vorticity located at the interface. This problem addresses the presence of inflection points in smooth profiles leading to ideal shear flow instabilities.

Consider an inviscid incompressible shear flow $\underline{\mathrm{V}} = \mathrm{Vy}(x)\hat{y}$ in a domain $0 \leq \times \leq a$, $-\infty < y < +\infty$. Show that for instability to occur, there must be at least one value of *x* in $[0,a]$ for which $\partial^2\mathrm{Vy}/\partial x^2 = 0$, i.e. there must be an inflection point in the flow. It is useful to approach this using the 2D vorticity advection equation and to write $\underline{\mathrm{v}} = \nabla\phi \times \hat{z}$. Also, write the frequency ω as $\omega_{\text{real}} + i\gamma$.

N.B.: The theorem you just derived was first proved by Rayleigh (who else?) and establishes only that an inflection point is necessary for instability. A second theorem, due to Fjortoft, demonstrates that a vorticity *maximum* is necessary.

5) Consider a thin extensible sheet that flexes itself in such a way that its coordinates $(x_s, y_s) = (x, a\ \sin(kx - \omega t))$, i.e. it oscillates in the vertical direction and a wave travels with velocity $c = \omega/k$ to the right. Such a motion is not time-reversible and we want to show that it results in a steady flow component U of the fluid above the sheet. The velocity can be calculated explicitly in the limit $\in = ka \ll 1$ and $U = \in^2 c/2$.

Introduce the stream function ψ such that the two components of the velocity $(u, v) = (\partial_y\psi, -\partial_x\psi)$. Consider the Stokes limit of small Reynolds numbers and show that the bi-Laplacian of ψ vanishes. Write down the boundary conditions dictated by the motion of the sheet $(x_s(t), y_s(t))$. Reduce the equations to a non-dimensional form by appropriate rescalings and assume $\in = ka$ small, i.e. deviations of the height of the free surface from $y = 0$ are small.

We shall seek a perturbative solution $\psi = \psi_0 + \in \psi_1 \ldots$. Expand in $\in$ the bi-Laplacian equation and the boundary conditions at the surface of the flexible sheet, and write down the corresponding equation and boundary conditions up to first order. Imposing the boundary condition that the flow stay finite as $y \to +\infty$, find the expression for ψ_0 and ψ_1 and verify that the latter tends to a constant as $y \to \infty$. Show that the constant coincides (in the original variables) with U. Acheson will be useful here.